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# TRIANGULAR FIN TEMPERATURE DISTRIBUTION BY THE METHOD OF DIFFERENTIAL QUADRATURE

Md. Moslemuddin Fakir[1]
Mahmood Khalid Mawlood[1]
Waqar Asrar[2]
Shahnor Basri[3]
Ashraf A. Omar[4]

Department of Aerospace Engineering
Faculty of Engineering
Universiti Putra Malaysia
43400 UPM, Serdang
Selangor, Malaysia

## ABSTRACT

*Differential Quadrature Method (DQM) is a numerical technique for the solution of initial and boundary value problems. This method has been successfully employed to solve a variety of physical science and engineering problems. In this paper, the solution of the heat distribution in a triangular fin has been obtained by the Differential Quadrature Method. The numerical solution has been compared with the exact solution and it agrees very well. The results show that unequally spaced mesh points yield stable and accurate results. The numerical solution obtained by using equally spaced mesh points is less accurate and lead to instability once the number of grid points exceeds a critical value.*

## 1.0 INTRODUCTION

Differential Quadrature Method (DQM) was first proposed by Bellman et al. [1] who solved successfully initial and boundary value problems. Areas of the problems in which the applications of DQM may be found in the literature include fluid mechanics, bioscience, structural mechanics, transport processes, static aeroelasticity and lubrication mechanics. It has been found that the DQM has a better capability of producing highly accurate solutions within minimal computational effort. Presently, there are many numerical discretization methods. The common aspect of all these methods is that the discretized form of a Partial Differential Equation (PDE) only involves the functional values at mesh points inside the solution domain. For these reasons, some numerical methods such as

[1]Graduate Research Assistant
[2]Associate Professor. Currently with the Dept. of Mech. Eng., Int. Islamic University, Malaysia
[3]Professor
[4]Lecturer
Email: moslem00@yahoo.com and waqar@iiu.edu.my

finite difference method [2-5] and the global method of differential quadrature [6-11] can only be applied to regular domain problems. The coordinate transformation can be made by numerical grid generation technique [12-13] and the need for coordinate transformation technique for irregular domain problems is actually due to the fact that the discretized form of a PDE only involves the mesh points inside the solution domain. Therefore, the physical boundary must be one of the mesh lines. On the other hand, if the numerical discretization of the PDE is not restricted by the solution domain, then the complicated coordinate transformation technique can be disregarded. Thus, the numerical computation for regular and irregular domain problems can be solved effectively in the Cartesian coordinate system. In this paper, we have applied the differential quadrature method (DQM) to the problem of heat transfer from a triangular fin.

## 2.0 DIFFERENTIAL QUADRATURE (DQ) RULE FOR WEIGHTING COEFFICIENTS AND MESH POINTS

In this discretization rule, the differentiations in a PDE are discretized by the rule of DQ and the rule is discussed in this section. The DQ rule was proposed by Bellman et al. [1], following the idea of integral quadrature. It is known that the integral of a function over a closed interval can be approximated by a weighted linear sum of all the functional values in the integral domain. Therefore, inspired from this idea and following Bellman et al. [1], we approximated the differentiation of a function at a mesh point by a weighted linear sum of all the functional values in the total domain. The main procedure in the DQ rule is the determination of the weighting coefficients. DQ discretization for a smooth function $f(x)$ in its $n$th order differentiation with respect to $x$ at a mesh point $x_i$ is

$$f_x^n(x_i) = \sum_{k=1}^{N} w_{ij}^n \cdot f(x_k) \tag{1}$$

for $i = 1, 2, \ldots, N$;

where, $N$ is the number of the grid points in the $x$ direction, and $f_x^n(x_i)$ represents the $n$th order differentiation with $n = 1, 2, \ldots, N-1$.

The weighting coefficients of the first order differentiation are calculated by the DQ rule as

$$w_{ij}^1 = \frac{m^1(x_i)}{(x_i - x_j) \cdot m^1(x_j)} \quad ; \quad i, j = 1, 2, \ldots, N, \text{ while } j \neq i \tag{2}$$

where

$$m^1(x_k) = \prod_{j=1, j \neq k}^{N} (x_k - x_j) \tag{3}$$

The diagonal weighting coefficients of the first order differentiation $w_{ii}^{1}$ can be obtained from

$$w_{ii}^{1} = -\sum_{j=1, j\neq i}^{N} w_{ij}^{1} \tag{4}$$

The weighting coefficients for the second and higher order differentiations can be calculated from the following recurrence relationship as

$$w_{ij}^{n} = n \cdot (w_{ii}^{n-1} \cdot w_{ij}^{1} - \frac{w_{ij}^{n-1}}{x_i - x_j}) \tag{5}$$

for $i, j = 1, 2, \ldots, N$; but $j \neq i$, $n = 2, 3, \ldots, N-1$.

In the same way, the diagonal weighting coefficients $w_{ii}^{n}$ can be computed from the formula

$$w_{ii}^{n} = -\sum_{j=1, j\neq i}^{N} w_{ij}^{n} \tag{6}$$

Weighting coefficients for the second and higher order differentiations can be calculated from the above first order differentiation completely. Most of the solutions give better accuracy for unequal spacing points. For this reason, it requires that the mesh points to be clustered towards the boundary. Distribution of such mesh points is shown in Figure 1.

Here, equal and unequal spacing mesh points distribution [8] are given by the formula

$$x_i = \frac{i-1}{N-1} \tag{7}$$

and

$$x_i = \frac{1}{2}(1 - cos\frac{i-1}{N-1}\pi)L \text{ , respectively,} \tag{8}$$

where, $L$ is the length of the computational domain $0 \leq x \leq L$ ; $i = 1, 2, \ldots, N$;.

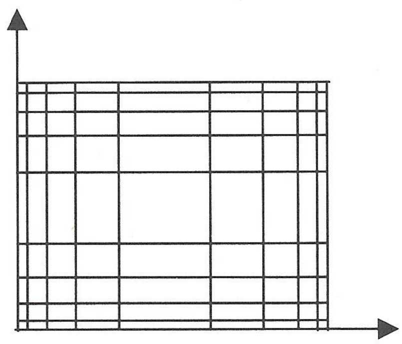

Figure 1 Typical non-uniform spacing mesh points used by DQ method

## 3.0 HEAT TRANSFER IN A TRIANGULAR FIN

One-dimensional thin triangular fin is considered in this paper. The fin is shown in Figure 2. Heat is transmitted along its length by conduction and dissipated from its lateral surfaces to the surroundings by convection. The governing equation for the temperature in the fin may be obtained by an energy balance and written in a dimensionless form [14] as

$$\varsigma \frac{d^2\theta}{d\varsigma^2} + \frac{d\theta}{d\varsigma} = m^2\theta, \qquad 0 \le \varsigma \le 1 \tag{9}$$

where, $\theta$ is the non-dimensional temperature i.e., $\theta = \frac{T}{T_{wall}}$ with $T$, the temperature at any point in the fin, $T_{wall}$, the temperature at the base of the fin; $\varsigma = \frac{x}{L}$ is the non-dimensional axial coordinate;

$m$ is given by $m^2 = \frac{hL^2}{k\delta}$, where $L$ and $\delta$ are the geometric parameters of the fin; $k$ and $h$ are the thermal conductivity and fin-to-ambient heat transfer coefficient, respectively.

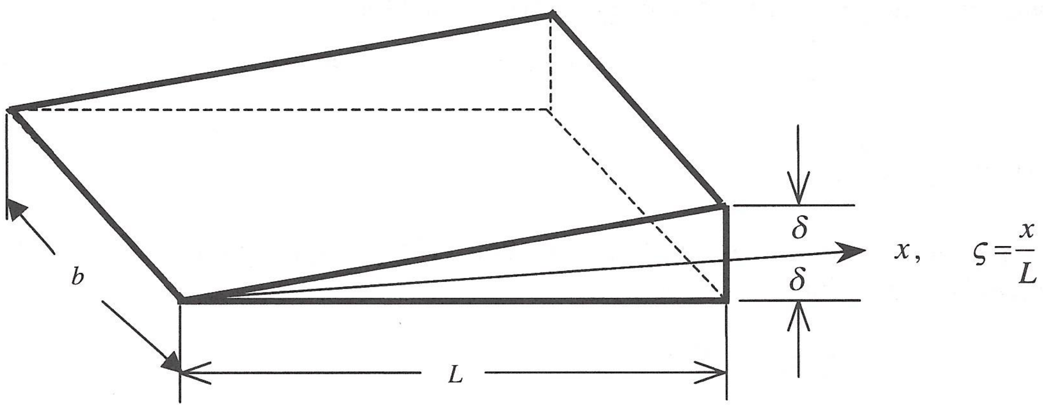


Figure 2 A triangular fin

Boundary conditions are given as follows:
The tip of the fin is insulated, i.e.,

$$\frac{d\theta}{d\varsigma} = 0 \qquad \text{at} \qquad \varsigma = 0 \tag{10}$$

and the base is at constant temperature, i.e.,

$$\theta = 1 \qquad \text{at} \qquad \varsigma = 1 \tag{11}$$

The exact solution is given in the form of the Bessel's function of the first kind with complex arguments [14]

$$\theta = \frac{J_0\left(2m\sqrt{-\varsigma}\right)}{J_0\left(2m\sqrt{-1}\right)} \tag{12}$$

For the quadrature solution of the system of equations (9) through (11), the requisite quadrature rules for the first and second order derivatives are given by

$$\frac{d\theta}{d\varsigma}\Big|_{\varsigma=\varsigma_i} = \sum_{j=1}^{N} w_{ij}^{1}\theta_j \tag{13}$$

and

$$\frac{d^2\theta}{d\varsigma^2}\Big|_{\varsigma=\varsigma_i} = \sum_{j=1}^{N} w_{ij}^{2}\theta_j, \qquad i = 1, 2, \ldots, N \tag{14}$$

where $N$ is the number of mesh points in the domain $0 \le \varsigma \le 1$.
Substituting the quadrature rules (13) and (14) in equation (9), the quadrature analog of the partial differential equation takes the form

$$\sum_{j=1}^{N}\left[\varsigma_i w_{ij}^2 + w_{ij}^1\right]\theta_j - m^2\theta_i = 0, \qquad i = 1, 2, \ldots, N \tag{15}$$

where $\theta_j = \theta(\varsigma_j)$

Using the quadrature rule (13) in equation (10), the quadrature analog of the boundary condition at $\varsigma = 0$ is

$$\sum_{j=1}^{N} w_{ij}^{1}\theta_j = 0, \qquad i = 1 \tag{16}$$

Also, the quadrature analog of the boundary condition at $\varsigma = 1$ as

$$\theta_i = 1, \qquad \text{for } i = N \tag{17}$$

Applying the fixed boundary condition, equation (17) in the quadrature analog of the partial differential equation (15), we obtain

$$\sum_{j=1}^{N-1}\left[\varsigma_i w_{ij}^2 + w_{ij}^1\right]\theta_j + \left[\varsigma_i w_{iN}^2 + w_{iN}^1\right]\theta_N - m^2\theta_i = 0$$

or, $\sum_{j=1}^{N-1}\left[\varsigma_i w_{ij}^2 + w_{ij}^1\right]\theta_j - m^2\theta_i = -\varsigma_i w_{iN}^2 - w_{iN}^1$ since $\theta_N = 1$, and $i$ = 2, 3, ..., $N$ - $1$

(18)

Similarly, using equation (17) in the quadrature boundary analog at the tip of the fin given in equation (16), we obtain

$$\sum_{j=1}^{n-1} w_{ij}^1 \theta_j = -w_{iN}^1 \,, \quad \theta_N = 1 \quad i = 1 \tag{19}$$

Also, equation (17) can be rewritten as

$$\theta_N = 1 \tag{20}$$

Finally, we obtain three linear algebraic equations (18), (19) and (20) for our numerical solution. These equations (18) to (20) are solved numerically using the Gaussian elimination algorithm. The results of the thin triangular fin problem are obtained for chosen values of the parameters m = 1 and $T_{amb} = 1$ for simplicity and are shown in Table 1 and Table 2. We have used here both equally and unequally spaced mesh points as shown in equations (7) and (8) respectively, i.e.

Type I: Equally spaced mesh points distribution

$$\varsigma_i = \frac{i-1}{N-1}, \qquad i = 1, 2, \ldots, N$$

Type II: Unequally spaced mesh points distribution ($L$ = 1)

$$\varsigma_i = \frac{1}{2}\left[1 - \cos\frac{i-1}{N-1}\pi\right]; \qquad i = 1, 2, \ldots, N$$

Once the numerical results at each mesh point are obtained, the relative percentage error are computed as

$$\text{Relative \% error} = \left|\frac{\theta_{numerical} - \theta_{exact}}{\theta_{exact}}\right| \times 100$$

## 4.0 RESULTS AND DISCUSSION

Since all material and geometric properties of the fin are represented by the parameter m, a representative value of m = 1 is chosen to study the numerical technique. Similar results may be easily obtained for different values of the parameter m. In Table 1 and Table 2, the maximum relative percent errors are shown at an interval of $\bullet \varsigma = 0.1$ for equally and unequally spaced mesh points respectively using cubic spline interpolation. The length of the solution domain is $0 \le \varsigma \le 1$ along the length of the fin. It is seen that the quadrature solutions are of very good accuracy when compared to the exact solution. The maximum error in the quadrature solution is at $\varsigma = 0$, i.e., at the tip of the triangular fin due to initialization and instability problem at the beginning and prominent with unequally spaced mesh points. With the increase of the value of $\varsigma$, errors are gradually reduced and at $\varsigma = 0.9$, the error is minimum. But, at $\varsigma = 1.0$, that is at the base of the fin, the error is zero due to the boundary condition, as numerical and exact values are the same there.

The convergence of the DQ solution for equal and unequal spacing mesh points have been compared and shown in Figure 3. It is apparent that the quadrature solution yields result with higher accuracy, of one order of magnitude or more, with unequally spaced mesh points as compared to that with equally spaced mesh points and this accuracy is prominent with relatively lower and higher values of the number of sampling points. It happens due to the mesh point distribution strategy of equally spaced and unequally spaced sampling points.

For equal spacing mesh points (Figure 3), the solution converges up to sampling points N = 45, from N = 46 through 51 the solution oscillates and after N = 51, the solution deteriorates very rapidly. In this case at N = 45, the best stable result is obtained. The reason for the deterioration of the results is due to the formation of ill-conditioned coefficient matrix in the numerical solution. As the number of sampling points increases, the errors increase in the computational domain to calculate the weighting coefficients, and after a certain value of sampling points, N, the weighting coefficients and hence the coefficient matrix became ill-conditioned. For this reason the solution deteriorates and the error increases rapidly with increasing N. However, on the contrary, the solution with unequally spaced sampling points (Figure 3) shows a monotonic convergence with increasing number of sampling points (shown up to N = 100). In addition, for the case of unequally spaced sampling points, excellent convergence is achievable for N > 31, whereas the average convergence range with equally spaced case is only $40 \le N \le 45$. This shows that for differential quadrature method, the solution for unequal spacing sampling point is better than those of equal spacing sampling points.

Figure 4 depicts the comparison of numerical and exact temperature distributions (results) and the corresponding percentage error for equally spaced sampling points with N = 45. The DQ numerical solution is very close to exact solution except at the tip of the fin. At $\varsigma = 0$, that is at the tip of the fin, it is observed that the amount of error is about 2.25 percent, which is very small and the error is gradually reduced with the increase of $\varsigma$ along the length of the fin. However, for unequally spaced sampling points with N = 45, the same comparison of fin temperature (exact, numerical) and corresponding percentage error of DQ solution are shown in Figure 5. Here, the maximum error is at the tip of the triangular fin and gradually reduced with the increase of $\varsigma$, similar to Figure 4. But in this case, the amount of error at the tip of the fin is only 0.2 percent, smaller than that with equal spacing sampling points. Since the DQ numerical solutions have been compared with exact solutions, the accuracy of our DQ numerical solution is clearly apparent here.

In Figure 6, the maximum absolute percentage errors of the DQ solution for equal and unequal spacing sampling points with N = 45 are shown. At $\varsigma = 0$, the errors are maximum as usual, which are 2.25 percent and 0.2 percent for equal and unequal spacing sampling points respectively. In both cases, errors converge smoothly with the increase of $\varsigma$, whereas unequal spacing shows better convergence compared to equal spacing throughout the range of $\varsigma$.

Investigating the various mesh points distribution for equal and unequal spacing, the unequally spaced mesh points distribution give better and accurate results and the solution converge smoothly with increasing number of mesh points.

## 5.0 CONCLUSION

The solution of the temperature distribution in a triangular fin is obtained using the method of Differential Quadrature. The results agree very well with the exact solutions and show the efficiency of the method. This method is more appropriate for unequally spaced mesh point distribution than those of equally spaced case. The solution obtained using equally spaced mesh points suffer from numerical instability while the unequally spaced mesh points provide a robust method for accurate solutions.

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Table 1 Solution with equally spaced mesh points

| $\varsigma$ | Exact Value | Maximum % error in quadrature solution | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | Number of sampling points (N) | | | | | | |
| | $\theta$ | 11 | 20 | 31 | 40 | 45 | 51 | 55 |
| 0.0 | 0.438 | 9.543 | 5.164 | 3.318 | 2.581 | 2.253 | 4.204 | 95.299 |
| 0.1 | 0.483 | 3.863 | 1.685 | 0.951 | 0.693 | 0.581 | 1.199 | 28.403 |
| 0.2 | 0.531 | 2.198 | 0.975 | 0.546 | 0.339 | 0.332 | 0.692 | 16.217 |
| 0.3 | 0.580 | 1.401 | 0.628 | 0.349 | 0.256 | 0.211 | 0.446 | 10.306 |
| 0.4 | 0.632 | 0.926 | 0.421 | 0.232 | 0.171 | 0.139 | 0.300 | 6.793 |
| 0.5 | 0.687 | 0.615 | 0.286 | 0.156 | 0.116 | 0.092 | 0.204 | 4.492 |
| 0.6 | 0.744 | 0.399 | 0.192 | 0.103 | 0.077 | 0.059 | 0.138 | 2.896 |
| 0.7 | 0.803 | 0.243 | 0.124 | 0.648 | 0.049 | 0.036 | 0.090 | 1.743 |
| 0.8 | 0.866 | 0.128 | 0.074 | 0.363 | 0.029 | 0.018 | 0.054 | 0.885 |
| 0.9 | 0.931 | 0.038 | 0.036 | 0.146 | 0.013 | 0.005 | 0.027 | 0.233 |
| 1.0 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 2 Solution with unequally spaced mesh points

| $\varsigma$ | Exact Value | Maximum % error in quadrature solution | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | Number of sampling points (N) | | | | | | |
| | $\theta$ | 11 | 20 | 31 | 40 | 45 | 51 | 100 |
| 0.0 | 0.438 | 3.022 | 1.002 | 0.451 | 0.283 | 0.229 | 0.182 | 0.053 |
| 0.1 | 0.483 | 0.645 | 0.167 | 0.068 | 0.040 | 0.032 | 0.024 | 0.006 |
| 0.2 | 0.530 | 0.386 | 0.098 | 0.039 | 0.023 | 0.018 | 0.014 | 0.003 |
| 0.3 | 0.580 | 0.207 | 0.064 | 0.025 | 0.015 | 0.011 | 0.009 | 0.002 |
| 0.4 | 0.632 | 0.164 | 0.042 | 0.017 | 0.010 | 0.007 | 0.006 | 0.001 |
| 0.5 | 0.687 | 0.109 | 0.028 | 0.011 | 0.006 | 0.005 | 0.004 | 0.001 |
| 0.6 | 0.744 | 0.068 | 0.019 | 0.007 | 0.004 | 0.003 | 0.002 | 0.000 |
| 0.7 | 0.803 | 0.045 | 0.011 | 0.004 | 0.002 | 0.002 | 0.001 | 0.000 |
| 0.8 | 0.866 | 0.028 | 0.006 | 0.002 | 0.001 | 0.001 | 0.001 | 0.000 |
| 0.9 | 0.931 | 0.010 | 0.003 | 0.001 | 0.007 | 0.000 | 0.000 | 0.000 |
| 1.0 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

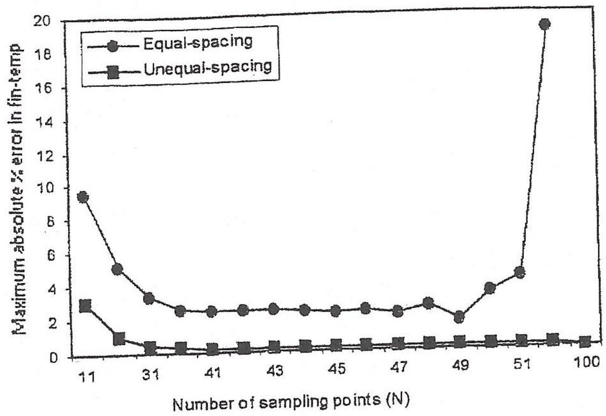


Figure 3 Comparison of convergence of fin-temperature for equal and unequal spacing of the DQ solution

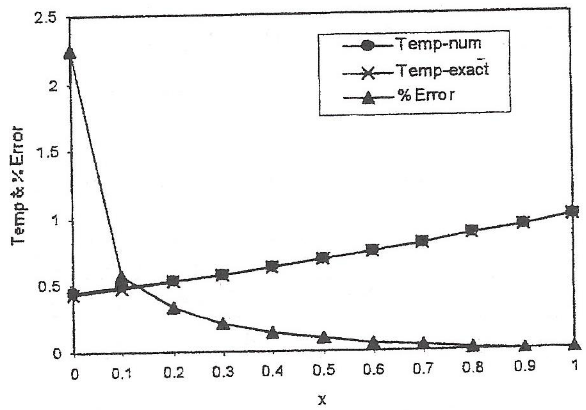


Figure 4 Comparison of fin-temperature (exact and numerical) of the DQ solution using equal-spacing and sampling points N = 45

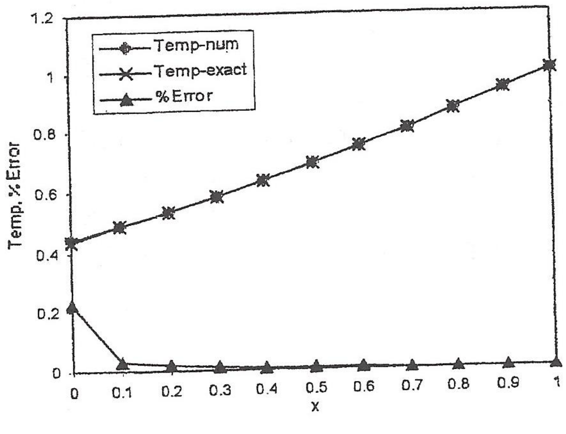


Figure 5 Comparison of fin-temperature (exact and numerical) of the DQ solution using unequal-spacing and sampling points N = 45

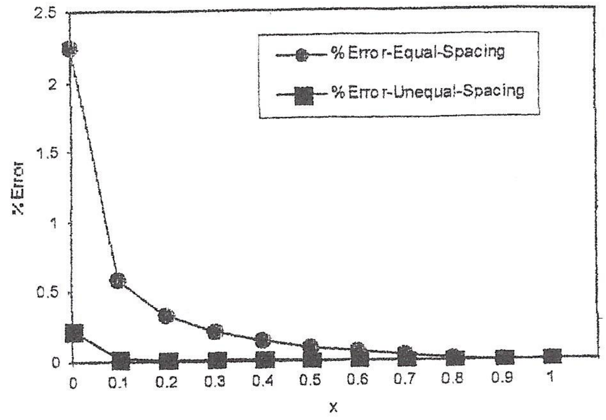


Figure 6 Error comparison of fin-temperature for equal and unequal spacing of the DQ solution with sampling points N,= 45