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A COMPARISON OF HIGHER-ORDER COMPACT FINITE-DIFFERENCE SCHEMES

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ABSTRACT

In this work, three fourth-order-accurate, compact finite-difference schemes, namely, the Hermitian (H), the cell-centered Hermitian (CCH) and the Spotz-Carey (S-C) schemes are studied. The schemes are described and their accuracies are investigated using the one-dimensional viscous Burgers equation as the testing model. Comparisons for the absolute, the average and the maximum relative errors are shown. The effect of spatial step size, h, on the accuracy of the selected schemes is investigated. A new procedure, for using the CCH scheme is proposed and found to produce the least error. The new procedure utilizes a combination of a fifth- and a sixth-order interpolation schemes. Other properties of the schemes, such as additional relations required and ease of implementation are also discussed.

Keywords: *High-Order Finite Difference, Burgers Equation*

1.0 INTRODUCTION

High-order compact finite difference schemes have attracted the attention of researchers due to their advantages over the traditional high order schemes. Many research papers on developing and implementing such schemes have appeared in the literature over the last 25 years.

Hirsh [1] has conducted numerical experiments with a class of $O(h^4)$ accurate compact schemes. The idea behind his scheme is that the derivatives are treated as unknowns at each point of the computational grid. Thus, for a second order differential equation, a system of two high-order relations, known as Hermitian relations [2] are used to evaluate the derivatives. Using the same approach, Adam [2] has proposed the elimination of the second derivative either implicitly by using the governing equation or explicitly by using another compact relation between the first and the second derivatives.

High-order compact relations similar to those used by Hirsh [1] and Adam [2], have been derived by Rubin and Khosla [3], and Goedheer and Potters [4] for non-uniform meshes. Lele [5] has presented and analyzed more generalized forms of the Hermitian schemes and introduced the notion of resolution efficiency as a measure of accuracy. Lele [5] has also developed compact schemes for the approximation of the first and the second derivatives on a cell-centered mesh, and has shown that the latter schemes have better resolution characteristics.

Schemes underlying the Hermitian approach; have been utilized by many researchers such as Cockburn and Shu [6], Haras and Ta'asan [7], Deng and Maekawa [8], Ravichandran [9] and Fu and Ma [10]. The most recent papers of Visbal and Gaitonde [11], Wilson et al [12], Ekaterinaris [13] and Reuter and Rempfer [14] are also based on this approach.

A different approach for developing high-order compact schemes has been presented by Spotz and Carey [15] and Spotz [16]. In this approach the governing equation is utilized to approximate the leading truncation error terms of the standard $O(h^2)$ central difference scheme. The resulting relation is then differenced compactly and included in the finite-difference formulation. This approach has been developed by MacKinnon and Carey [17] and Abarbanel and Kumar [18] independently about the same time. Similar schemes were proposed and used by Dennis and Hudson [19], Gupta et al [20] and Asrar et al [21].

It can be noted that all the high-order schemes cited in this review may be grouped under two general approaches, namely, the Hermitian approach and the Spotz-Carey approach. The objective of the present paper is to compare the accuracy of the different $O(h^4)$ accurate compact schemes underlying these two approaches.

The non-dimensionalized non-linear Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad ; \quad -9 \le x \le +9 \tag{1}$$

together with its analytical solution [22]

$$u = \frac{-2\sinh x}{\cosh x - e^{-t}} \tag{2}$$

is selected as the test problem.

Three $O(h^4)$ descretization methods are selected and used to approximate the space derivatives in Eq. (1), namely; 1- the original Hermitian scheme (H) as presented by Hirsh [1] and Adam [2], 2- the cell-centered Hermitian scheme (CCH) developed by Lele [5] and 3- the Spotz-Carey (S-C) scheme [13,14]. The solution procedure is advanced in time by a standard fourth order Runge-Kutta method.

In the following sections, descretization schemes are described; results and comparisons in terms of absolute, average and relative error are shown. The effect of spatial step size h, on the accuracy of the different schemes is also studied. Estimates of the spatial order of accuracy of the schemes are shown.

2.0 COMPACT $O(h^4)$ APPROXIMATIONS TO SPACE DERIVATIVES

2.1 The Hermitian (H) Scheme

In this scheme the governing equation (1) is approximated as

$$\frac{\partial u_i}{\partial t} + u_i F_i = S_i \tag{3}$$

where F_i and S_i are approximations to the first and the second derivatives respectively.

The fourth-order compact relations used to approximate F_i and S_i are [1,2]

$$\frac{1}{4}F_{i-1} + F_i + \frac{1}{4}F_{i+1} = \frac{3}{4h}(u_{i+1} - u_{i-1}) \tag{4}$$

$$\frac{1}{10}S_{i-1} + S_i + \frac{1}{10}S_{i+1} = \frac{6}{5h^2}(u_{i+1} - 2u_i + u_{i+1}) \tag{5}$$

These two relations yield tridiagonal systems of equations for F_i and S_i requiring two additional boundary conditions. Following Adam [2]; the approximations, S_i are eliminated by utilizing equation (3) to give the tridiagonal system

$$\frac{1}{12}\frac{\partial u_{i-1}}{\partial t} + \frac{\partial u_i}{\partial t} + \frac{1}{12}\frac{\partial u_{i+1}}{\partial t} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{1}{12}(u_{i+1}F_{i+1} + u_{i-1}F_{i-1}) - \frac{5}{6}u_i F_i \tag{6}$$

Thus only equation (4) is needed, with its boundary conditions, to be solved in addition to (6). Equation (6) can be solved for the time derivatives $\frac{\partial u_i}{\partial t}$, and the latter are then integrated in time.

2.2 The cell-centered (CCH) scheme

The most compact $O(h^4)$ approximation forms for the first and the second derivatives on a cell-centered mesh developed by Lele [5] are

$$\frac{1}{22}F_{i-1} + F_i + \frac{1}{22}F_{i-1} = \frac{24}{22h}(u_{i+1/2} - u_{i-1/2}) \tag{7}$$

$$\frac{1}{46}S_{i-1} + S_i + \frac{1}{46}S_{i+1} = \frac{96}{23h^2}(u_{i+1/2} - 2u_i + u_{i-1/2}) \tag{8}$$

It is necessary to use accurate interpolation schemes for the evaluation of the mid-point values of u in (7) and (8). A sixth-order compact interpolation scheme given by Lele [5], can be written as

$$\frac{3}{10}u_{i-1/2} + u_{i+1/2} + \frac{3}{10}u_{i+3/2} = \frac{1}{20}(u_{i+2} + u_{i-1}) + \frac{3}{4}(u_{i+1} + u_i) \tag{9}$$

In this study, however, unstable solution is observed when mid-point values obtained from (9) are used in (7) and (8). Fifth-order left- and right- biased interpolants used by Deng and Maekawa [8] are

$$\frac{1}{2}u^{-}_{i-1/2} + u^{-}_{i+1/2} + \frac{1}{10}u^{-}_{i+3/2} = \frac{1}{10}u_{i-1} + u_i + \frac{1}{2}u_{i+1} \tag{10a}$$

$$\frac{1}{10}u^{+}_{i-1/2} + u^{+}_{i+1/2} + \frac{1}{2}u_{i+3/2} = \frac{1}{2}u_i + u_{i+1} + \frac{1}{10}u_{i+2} \tag{10b}$$

In Ref. [8], these relations are used in Roe's approximate Riemann solvers for flux differencing. In the present work, the scheme (10) is utilized rather differently. It is proposed, first, to use the average of the left- and right-biased values in determining $u_{i+1/2}$, such that

$$u_{i+1/2} = \frac{1}{2}(u^{+}_{i+1/2} + u^{-}_{i-1/2}) \tag{10c}$$

The mid-point values obtained from (10c) can be used directly in the high-order relations (7) and (8). Alternatively, relation (8) can be eliminated by use of the governing equation (3). Significant improvement over the H scheme is observed in the results when relations (10) are used to approximate the derivatives. Next, it is proposed to use a combination of the schemes (9) and (10) and is found to produce striking results. The mid-point values obtained from (10c) are used to approximate the first derivatives, F_i and those obtained from (9) are used to approximate the second derivatives, S_i. Detailed results will be shown for the combined scheme.

2.3 The Spotz-Carey (S-C) Scheme

In this scheme, the space derivatives in (1) are central differenced to give

$$\frac{\partial u_i}{\partial t} + u_i\frac{u_{i+1} - u_{i-1}}{2h} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + u_i\frac{h^2}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_i - \frac{h^2}{12}\left.\frac{\partial^4 u}{\partial x^4}\right|_i - O(h^4) \tag{11}$$

The leading error terms are approximated to give an $O(h^4)$ method. This is accomplished [15,16] by differentiating (1) to give

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_i = \left[\frac{\partial^2 u}{\partial t \partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]_i, \tag{12}$$

which can be approximated as

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_i = \frac{\partial}{\partial t}\left(\frac{u_{i+1} - u_{i-1}}{h}\right) + \left(\frac{u_{i+1} - u_{i-1}}{2h}\right)^2 + u_i\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) \tag{13}$$

and similarly

$$\begin{aligned}\left.\frac{\partial^4 u}{\partial x^4}\right|_i &= \left[\frac{\partial^3 u}{\partial t \partial x^3} + 3\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}\right]_i \\ &= \frac{\partial}{\partial t}\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) + 3\left(\frac{u_{i+1} - u_{i-1}}{2h}\right)\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) + u\left.\frac{\partial^3 u}{\partial x^3}\right|_i\end{aligned} \tag{14}$$

Equations (14) and (13) can be combined with (11) to yield the tridiagonal system

$$a_i \frac{\partial u_{i-1}}{\partial t} + \frac{\partial u_i}{\partial t} + b_i \frac{\partial u_{i+1}}{\partial t} = \frac{6}{5} c_i, \tag{15}$$

where

$$a_i = \frac{1}{10} + \frac{h}{20} u_i \tag{16}$$

$$b_i = \frac{1}{10} - \frac{h}{20} u_i \tag{17}$$

and,

$$\begin{aligned} c_i = (u_{i+1} - 2u_i + u_{i-1})\left(\frac{1}{h^2} + \frac{u_i^2}{4} - \frac{u_{i+1} - u_{i-1}}{8h}\right) \\ + u_i\left(\frac{u_{i+1} - u_{i-1}}{2h}\right) + \frac{u_i}{48}(u_{i+1} - u_{i-1})^2 \end{aligned} \tag{18}$$

Thus, the system of equations (15) is solved for $\frac{\partial u_i}{\partial t}$ which are then integrated by the fourth-order Runge-Kutta method.

3.0 BOUNDARY AND INITIAL CONDITIONS

Boundary and near boundary grid points need special procedures as a result of using high-order compact schemes. One-sided second- and third-order relations for the first and second derivative approximations may be provided easily [1,2]. In the present study, however, in order to provide a sound basis for comparison between the different schemes, boundary values of u and its space and time derivatives are all assumed to be known and obtained by using Eq. (2). Boundary conditions for the mid-point interpolation schemes are provided using the following third-order relations [8],

$$u_{1+1/2} + \frac{1}{2}u_{2+1/2} = \frac{5}{16}u_1 + \frac{9}{8}u_2 + \frac{1}{16}u_3 \tag{19}$$

$$\frac{1}{2}u_{N-3/2} + u_{N-1/2} = \frac{1}{16}u_{N-2} + \frac{9}{8}u_{N-1} + \frac{5}{16}u_N \tag{20}$$

The initial profile of u is generated from Eq. (2) for $-9 \le x \le 9$ at time $t = 0.01$ such that a steep velocity gradient has appeared in the region close to $x = 0$.

4.0 DISCUSSION OF RESULTS

From stability constraints, the step sizes $h = 0.2$ and $\Delta t = 0.01$ are shown in [22] to produce stable results for Eq. (1) with different low order finite-difference methods. Numerical solutions of Eq. (1) obtained from the schemes H, CCH and S-C with these step sizes are plotted in Figs. 1(a)-1(c) respectively, as well as the analytical solution. The development of the velocity profile at four time levels, (t = 0.02, 0.1, 0.5, 1.0) are shown. All schemes exhibit stable solutions for the space and time step sizes selected.

A comparison of absolute errors produced by the three schemes at time levels (t = 0.02, 0.1,0.5) is shown in Figs. 2(a)-2(c). The absolute error is determined as the absolute value of the difference between the analytical and the numerical solutions. It is noted that the error produced by the CCH scheme is the lowest, while the H scheme produces the highest error. It is also noted that as time proceeds, the error produced by the different schemes smears spatially at different rates. The error due to the S-C scheme smears spatially more rapidly than the H and the CCH schemes. Perhaps, it may be more convenient to compare in terms of the average error, which represents the area under the absolute error curve for the entire spatial domain at a given time. Fig. 3 shows the development of the average errors with time for all the three schemes. Similar trends are exhibited by the maximum relative error as shown in Fig. 4. It appears that the CCH scheme provides the best results over the entire calculation period.

The effect of spatial step size h on the accuracy of the schemes is also investigated and results for the average error at time level t = 0.5 are shown in Fig. 5. The time step ($\Delta t = 0.01$) is kept constant for all these calculations. The order of accuracy of each of the schemes is estimated from the slope of the curve and is shown in the legend. It is observed that the CCH scheme exhibits highest order of accuracy and the S-C scheme shows the lowest.

In general, comparing the errors shows that the S-C scheme is better than the H scheme for the range of spatial step sizes shown in Fig. 5 and the CCH scheme is the best. However, if one considers the number of additional equations required by each scheme during the calculation process, the CCH scheme requires the maximum (5 as used here), the H scheme requires one, while the S-C scheme requires no additional equation. It must be noted that the additional required relations are tridiagonal algebraic systems that can be inverted easily. The cost of solving the additional equations may partially be paid by using a coarse grid. For example, if an average error of 0.001 is desired, it can be shown from Fig. 5 that the CCH scheme requires about 18 and 20% less grid points as compared to the S-C and the H schemes respectively. Moreover, the CCH schemes may provide results comparable with those of the S-C scheme with only 3 additional relations. This is achieved by using relations (10), as stated in Sec. 2.2, to evaluate the $F_i's$ and eliminating the $S_i's$. Results for the maximum relative error obtained by the latter method are shown on Fig. 4 as the curve designated by CCH1.

Another important aspect that must be considered is the simplicity of application. The H and the CCH schemes are easy to use and the high-order relations as well as the interpolation schemes will not change if a new problem is considered. On the other hand, the S-C scheme is algebraically involved and each new problem requires a whole new approximation procedure. For multi-dimensional problems the algebra required by the S-C scheme becomes prohibitively complicated.

From the above discussion, it may be concluded that the CCH scheme has preferable properties to the H and the S-C schemes.

5.0 CONCLUSIONS

The accuracy of three $O(h^4)$ compact schemes underlying two different approximation approaches is investigated. The one-dimensional viscous Burgers equation is used as a benchmark for testing and comparing the accuracies of the methods. The CCH method is found to produce the least error when used with a combination of a fifth-order and a sixth-order interpolation schemes. Other aspects, such as additional calculation procedures required by the schemes and ease of implementation are also discussed.

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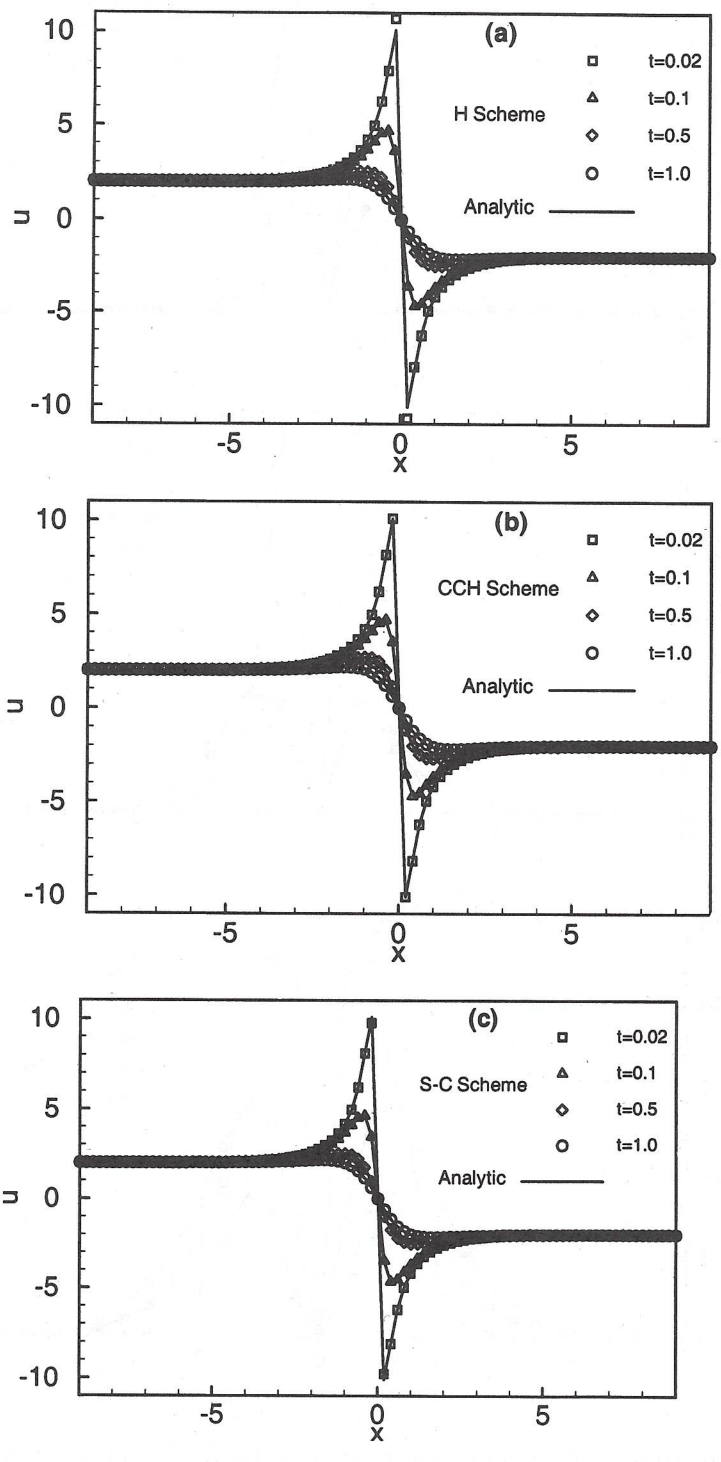

Figure 1 Solution of the Burgers equation by the different schemes

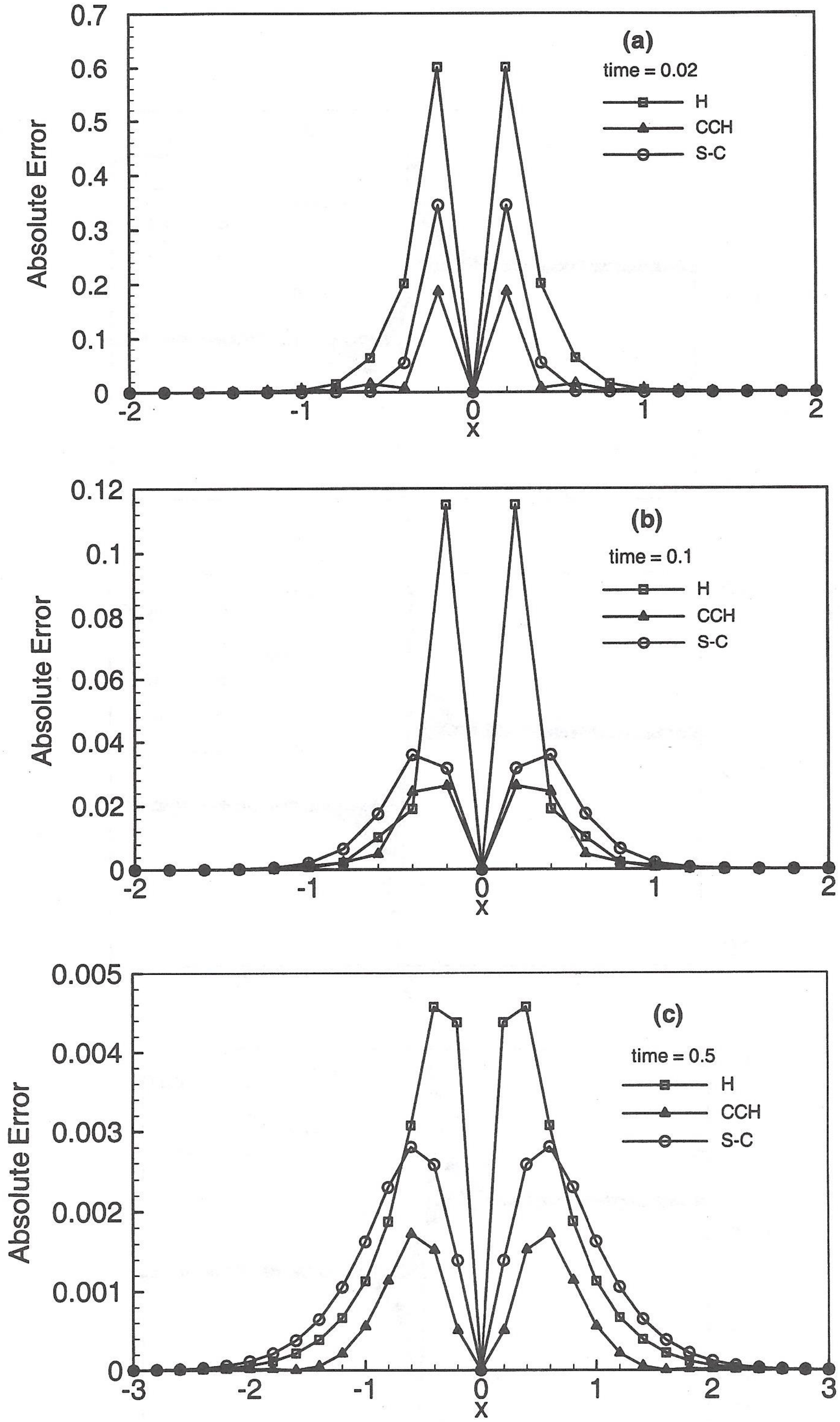

Figure 2 Comparison of the absolute error at a) t = 0.02, b) t = 0.1 and c) t = 0.5

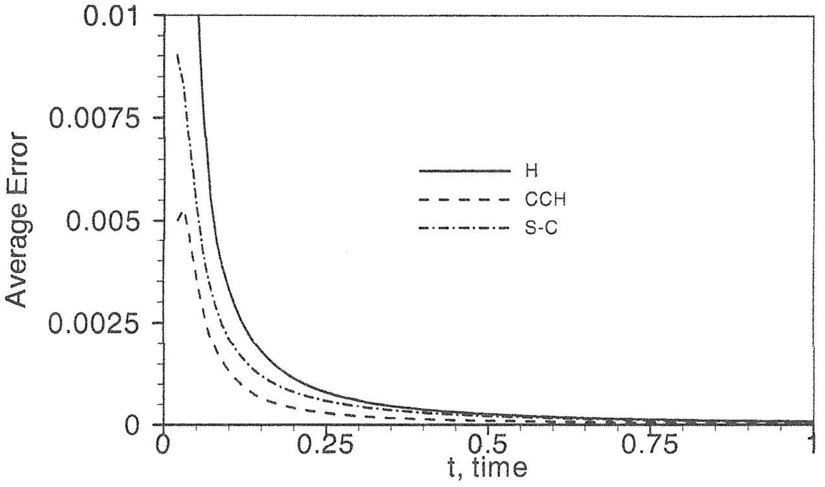

Figure 3 Comparison of the average error

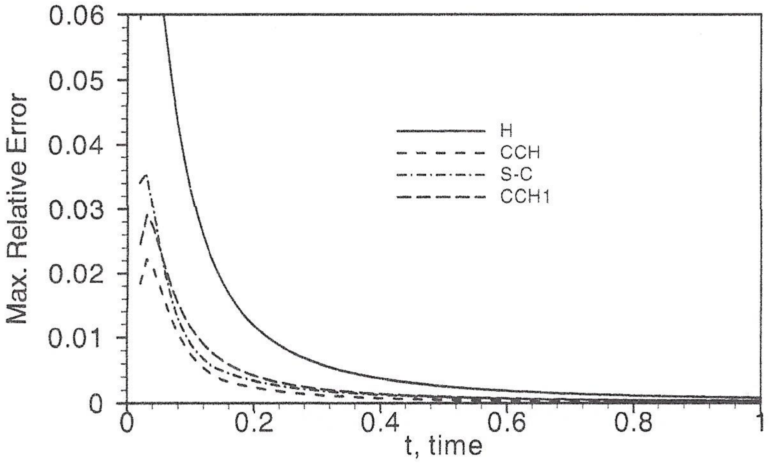

Figure 4 Comparison of the maximum relative error

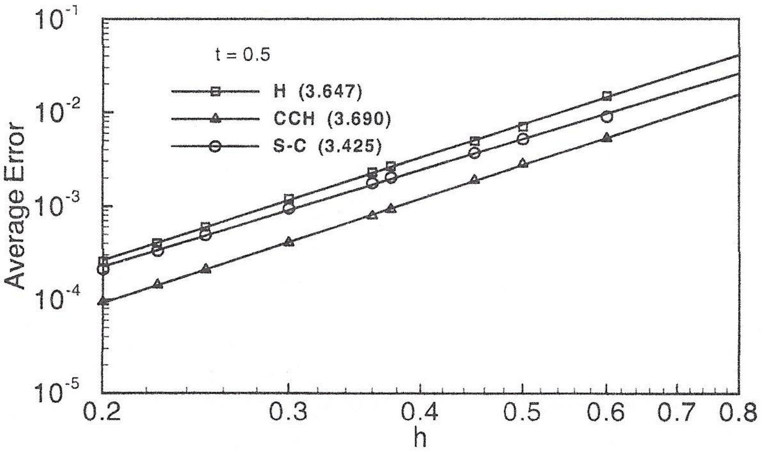

Figure 5 The Effect of step size h, on accuracy at time level t = 0.5