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FOURTH-ORDER ACCURATE FINITE DIFFERENCE SOLUTION OF THE TRANSIENT HEAT TRANSFER EQUATION

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ABSTRACT

In this paper a high-order compact solution is presented for the transient diffusion equation subject to both homogeneous as well as insulated boundary conditions. The finite difference scheme is fourth order accurate both in space and time. The central difference scheme is used to approximate the space derivatives. The higher order terms in the Taylor series expansion are approximated using the governing differential equations. The difference equations are integrated by applying a Runge-Kutta scheme. Numerical results are compared with exact solutions.

Keywords: *High-order compact finite difference scheme, Transient Diffusion, heat equation.*

1.0 INTRODUCTION

The finite difference method of solving partial differential equations involves discretizing the domain of interest into a grid on which a discrete approximation to the governing differential equation is applied. While the central difference scheme can give an $O(h^2)$ accurate result without any difficulty. In order to obtain results of higher accuracy the size of the mesh (i.e. h) needs to be reduced thereby increasing the size of the matrix. One can resort to $O(h^4)$ accurate formulation using the central difference scheme but this will increase the matrix density and bandwith and hence the cost of the computation. Also the approximations to the boundary conditions are more complicated. Methods, which have accuracy more than $O(h^2)$ are called higher-order methods. These methods are desirable because they allow for coarser meshes thereby reducing the computational effort.

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The usual procedure for obtaining higher-order accuracy is to include additional grid points into the approximations for the derivatives. These methods though of higher order yield non-compact stencils. They use grid points located beyond those directly adjacent to the node at which the derivative is being approximated. This increases the matrix bandwidth and complicates formulation near boundary nodes. Such methods have been widely used [1, 2, 3, 4, 5, 6, 7]. Compact formulations of higher-order methods are desirable because of reduced matrix bandwidth and ease in approximating derivatives at points near the boundaries. Such schemes were studied by several researchers [8, 9, 10, 11].

Abarbanel and Kumar [12] developed a High-Order Compact (HOC) scheme for Euler equations. Similar schemes have been proposed HOC schemes having fourth and sixth order accuracy for incompressible flows [13. 14. 15, 16, 17, 18].

In this paper the transient heat diffusion equation in the one-dimensional form is considered. The scheme used in this paper increases the accuracy of the usual central difference scheme from $O(h^2)$ to $O(h^4)$ by including compact approximations of the truncation error terms. It is based on an idea of MacKinnon and Carey [21]. Both Dirichlet and Neumann boundary conditions are applied. The technique developed by Spotz [19, 20] is used to approximate the space derivatives. This technique uses the Crank-Nicolson method to discretize the time derivatives resulting in a second order accurate formulation in time, an effort has been made to use the fourth order Runge-Kutta method to integrate the difference equations. This increases the accuracy to fourth order in the time domain as well, resulting in an overall accuracy of fourth order in space and time.

2.0 PROBLEM STATEMENT

We consider the standard one dimensional transient linear heat conduction equation [22] with constant material properties.

$$u_t = c^2 u_{xx} \tag{1}$$

Where u is the dimensionless temperature, c^2 is thermal diffusivity, the subscript t refers to partial derivatives with respect to time and the subscript x refers to partial derivatives with respect to the space variable x.

Along with the homogeneous conditions

$$u(0, t) = 0, u(L, t) = 0 \tag{2}$$

and initial condition:

$$u(x, 0) = g(x) \tag{3}$$

Equation (1) represents a model of the physical problem shown in Figure 1.

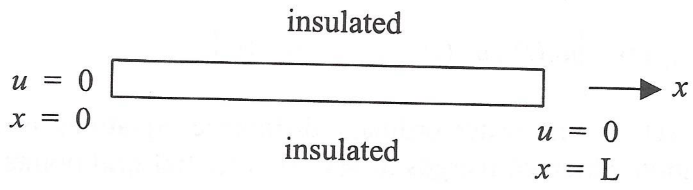

Figure 1 One-dimensional heat conduction, conducting boundaries

2.1 High Order Compact Formulation

Substituting central differences into Equation (1) we get :

$$u_{t,\,i} = \frac{c^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) - \frac{c^2h^2}{12}u_{xxxx,\,j} + O(h^4) \tag{4}$$

where h is the increment the subscript i refers to the nodal position in the x-direction, and the subscript j is an index for time.

Differentiate Equation (1), we obtain:

$$u_{xxxx,\,j} = \frac{1}{c^2}u_{txx} \tag{5}$$

Substituting Equation (5) into Equation (4) and rearranging it can be written as:

$$u_{t,\,i} + \frac{h^2}{12}u_{txx} = \frac{c^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + O(h^4) \tag{6}$$

Substituting the standard $O(h^2)$ central difference approximation to u_{xx} in Equation (6) we obtain:

$$\frac{d}{dt}u_{i,j} + \frac{1}{12}\frac{d}{dt}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) = \frac{c^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + O\left(h^4\right) \tag{7}$$

Rewriting Equation (7) we get:

$$\frac{d}{dt}\left(u_{i-1,j} + 10u_{i,j} + u_{i+1,j}\right) = \frac{12c^2}{h^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i-1,j}\right) + O\left(h^4\right) \tag{8}$$

The semi-discretized version of the fourth order accurate Equation (8) is;

$$A\frac{du_i}{dt}(t) = \frac{12c^2}{h^2}B\,u_i(t)\,,\; i = 0, 1, \ldots\ldots, N,\; t \geq 0 \tag{9}$$

where:

$$u_i\,(t) = \left[u_0(t)\,,\,u_1\,(t)\,,\,.......,\quad u_N\,(t)\right] \tag{10}$$

Equation (9), a set of first order ordinary difference equations is the semi-discrete approximation which converges to $u(x_i,\ t)$ at spatial grid points x_I, and A and B are coefficient matrices.

3.0 HOMOGENEOUS BOUNDARY CONDITIONS

For a bar of length, L, with homogeneous boundary conditions $u\ (0,\ t) = u\ (L,\ t) = 0$ we get:

$$u_O\,(t) = u\,(0,t) = 0 \tag{11}$$

$$u_N\,(t)\ =\ u\,(L,t) = 0 \tag{12}$$

Substituting the boundary conditions (11) and (12) into Equation (9) we can obtain the coefficient matrices, A and B, given as:

$$A = \begin{bmatrix} 10 & 1 & 0 & 0 & .. & 00 & 0 \\ 1 & 10 & 1 & 0 & .. & 00 & 0 \\ 0 & 1 & 10 & 1 & .. & 00 & 0 \\ . & & & & & & . \\ 0 & 0 & 0 & 0 & .. & 110 & 1 \\ 0 & 0 & 0 & 0 & .. & 01 & 10 \end{bmatrix} \tag{13}$$

$$B = \begin{bmatrix} -2 & 1 & 0 & 0 & .. & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & .. & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & .. & 0 & 0 & 0 \\ . & & & & & & & . \\ 0 & 0 & 0 & 0 & .. & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & .. & 0 & 1 & -2 \end{bmatrix} \tag{14}$$

Since det $(A)\ \neq 0$, A is invertible and Equation (9) can be written as

$$\frac{du_i(t)}{dt} = \frac{12c^2}{h^2} A^{-1} B\, U_i(t) \tag{15}$$

With initial condition:

$$u_i(0) = g(x_i) \tag{16}$$

Any standard Runge-Kutta integrator can be applied to Equation (15) with the initial conditions given by Equation (16).

The exact solution for Equation (1) subject to the boundary condition (2) with the initial condition:

$$g(x) = \sin\left(\frac{\pi x}{L}\right) \tag{17}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} An \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda^2 nt} \tag{18}$$

where

$$A_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) \lambda_n = \frac{cn\lambda}{L} \tag{19}$$

and

$$\lambda_n = \frac{cn\lambda}{L}$$

3.1 Numerical Results

For $h = \frac{1}{6}$, c = 1, k = 0.01 and L = 1, numerical results are obtained. Here MATLAB is used to calculate the results. For time step $k = 0.01$ the Euclidean norm of the error obtained from MATLAB is shown in Table 1.

Table 1 Norm of the error vector for homogeneous boundary condition

T	Norm of error vector
0	0
1.000000000000000E-002	4.902156805896193E-005
2.000000000000000E -002	8.882942012123435E-005
3.000000000000000E -002	1.207233886443459E-004
4.000000000000000E-002	1.458397143930254E-004
5.000000000000000E-002	1.651705518284252E-004
6.000000000000000E-002	1.795831431373788E-004
7.000000000000000E-002	1.898253186337675E-004
8.000000000000000E-002	1.965583787824429E-004
9.000000000000000E-002	2.003495283676112E-004
1.000000000000000E-001	2.016931803852542E-004

3.2 Discussion

The high-order compact scheme presented here has an error of $O(h^4, k^4)$. With h selected as 1.6667 e –01 and k given the value 0.01 the error should be proportional to the order of the error in the scheme. The numerical results shown in Table 1 prove that the error is indeed proportional to h^4 which is the dominating quantity in the error term.

4.0 INSULATED BOUNDARY CONDITION

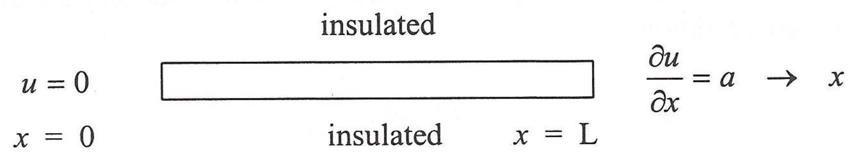

Figure 2 One-dimensional heat conduction, conducting and insulated boundaries

Next we consider the bar to be insulated at one end. Hence for Equation (1) the boundary condition (2) becomes

$$u\,(0, t) = 0, \quad \frac{\partial u(L, t)}{\partial x} = a \tag{20}$$

where a is a constant.

Implementing the Neumann Boundary condition presents no difficulty and is done as in the homogeneous boundary conditions. To implement the boundary condition (20) we need a $0(h^4)$ accurate approximation to the first derivative using backward difference scheme as follows:

$$u'_i = \frac{u_i - u_{i-1}}{h} + \frac{h}{2} u''_i - \frac{h^2}{6} u'''_i + \frac{h^3}{24} u_i^{iv} + O(h^4) \tag{21}$$

Where all derivatives are evaluated at $x = L$ or $i = N$ and from Equation (1)

$$u'' = u_{xx} = \frac{1}{c^2} u_t \tag{22}$$

then

$$u''' = u_{xxx} = \frac{1}{c^2} u_{tx} = \frac{1}{c^2} \frac{d}{dt}(\alpha) = 0 \tag{23}$$

and

$$u^{iv} = u_{xxxx} = \frac{1}{c^2} u_{txx} = \frac{1}{c^2} \frac{\partial}{\partial t}(u_{xx}) \tag{24}$$

An $O(h)$ accurate approximation for u'' (using backward difference scheme) is:

$$u'' = u_{xx} = \frac{2}{h^2}(u_{i-1} - u_i) + O(h) \tag{25}$$

Using Equations (23) and (25 – 28) in Equation (24) and rearranging the $O(h^4)$ accurate approximation (24) becomes

$$\frac{d}{dt}\left(u_{N-1} + 5u_N\right)_j = \frac{12c^2}{h^2}\left(u_{i-1} - u_i + ah\right)_j + 0(h^4) \tag{26}$$

Equation (29) is applied at the boundary node $x = L$ or $i = N$ to yield:

$$\frac{d}{dt}\left(u_{N-1} + 5u_N\right)_j = \frac{12c^2}{h^2}\left(u_{N-1} - u_N + ah\right)_j \tag{27}$$

Now in this case the semi-discretized Equation (9) can be written as:

$$A\frac{d}{dt}U_i(t) = \frac{12c^2}{h^2}\left[B\left[u_i(t)\right] + C\right] \tag{28}$$

where the matrices A, B and C are defined as:

$$A = \begin{bmatrix} 10 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 10 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 10 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 5 \end{bmatrix} \tag{29}$$

$$B = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -1 \end{bmatrix} \tag{30}$$

and

$$C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ ah \end{bmatrix} \tag{31}$$

Matrices A, B and C are obtained by substituting the boundary conditions in Equation (28) and writing it for the N node points.

Equation (31) may be rewritten as,

$$\frac{d}{dt}u_i(t) = \frac{12c^2}{h^2}\left(A^{-1}Bu_i(t) + A^{-1}C\right) \quad (32)$$

Equation (32) together with the initial condition (16) can be integrated by a standard Runge-Kutta integrator.

The exact solution of Equation (1) subject to the boundary condition (23) with $a = 0$ and the initial condition

$$u(x,0) = g(x) = \sin\left(\frac{\pi x}{2L}\right) \quad (33)$$

is given by:

$$u(x,t) = \sin\left(\frac{\lambda x}{2L}\right) e^{-\frac{\pi^2 c^2 t}{4L^2}} \quad (34)$$

4.1 Numerical Results

For a bar of length, $L = 1$, for totally insulated at one end, i.e, $a = 0$, h = 1/6, $k = 0.01$ and $c = 1$, the numerical solution is obtained using MATLAB. For time step $k = 0.01$ the Euclidean norm of the error between the numerical and the exact solutions is shown in Table 2.

Table 2 Norm of the error vector for adiabatic boundary condition

T	Norm of error vector
0	0
1.000000000000000E-002	8.838840899562461E-007
2.000000000000000E -002	1.724684381933329E-006
3.000000000000000E -002	2.523975924440418E-006
4.000000000000000E-002	3.283282583385749E-006
5.000000000000000E-002	4.004078601751526E-006
6.000000000000000E-002	4.687790112110461E-006
7.000000000000000E-002	5.335796607678521E-006
8.000000000000000E-002	5.335796607678521E-006
9.000000000000000E-002	5.335796607678521E-006
1.000000000000000E-001	5.335796607678521E-006

4.2 Discussion

As in the previous case of homogeneous boundary conditions, the error in the case of adiabatic boundary conditions should also be proportional to $O(h^4,k^4)$ implying that this is a fourth-order accurate scheme. The numerical computations done by integrating in time with a fourth-order Runge-Kutta procedure using

MATLAB and computing the Euclidean norm of the error vector using the exact solution indeed prove the assumption that the error is proportional to fourth-order. Since the increment in the space direction is much larger than the increment in time, the error is dominated by h. This is shown in Table 2 where the norm is presented at various times.

5.0 CONCLUSION

A fourth order accurate solution in space and time has been obtained for the one-dimensional transient heat transfer equation subjected to homogeneous as well as mixed boundary conditions. The numerical solution agrees extremely well with the exact solution as shown by the magnitude of the norm of the absolute error vector at various times. The error in case of the homogeneous boundary condition as well as in the case of the mixed boundary conditions are proportional to the fourth power of the increments in space and time

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